EGOI 2024 Editorial - Infinite Race

Problem author: Nils Gustafsson.

The problem

Anika is one of the N participants (numbered from 0 to $N-1$) in a race around a circular track. She is participant 0 and starts right after the finish line with all other participants positioned ahead of her on the track. Anika cannot keep track of how many laps she has run, but she remembers when she overtakes someone or when someone overtakes her. What is the minimum number of times she must have crossed the finish line? Nobody moves backwards, and no overtaking happens exactly at the finish line. Furthermore, note that the participants do not necessarily run at a constant speed.

Test group 1: $N = 2$

In this test group there is just one other participant.

First, why is the answer not always 0, in other words how can we be sure that Anika has crossed the finish line? In case she overtakes the other participant twice in a row, since the other participant does not run backwards, Anika must have completed a full lap in between and has therefore crossed the finish line at least once.

It turns out that this is the only interesting case. More specifically, the answer is equal to the number of pairs of consecutive events in the input that both correspond to Anika overtaking the other participant (denoted as 1 in the input format).

It is clear that this is a lower bound on the answer. In order to see how we can achieve exactly this many crossings of the finish line, assume that both Anika and the other participant are running very slowly unless they need to overtake, and carefully consider whether Anika or the other participant is ahead within the lap (in other words, whether we first encounter Anika or the other participant if we go from the finish line along the track).

When Anika overtakes the other participant (event 1), she is now ahead of them, and in case she was also ahead before, we add one more finish line crossing to the answer. When the other participant overtakes Anika (event -1), the other participant is now ahead, so there will be no finish line crossing on the next event 1.

Test group 2: Anika only overtakes

In this test group in all events Anika overtakes other participants.

We can assume the other participants are running so slowly that they are effectively standing still, and simply count how many times did she overtake each of the other participants, and find the maximum of those counts. This number minus one will be the answer.

Test group 3: general case, few events

In this test group all types of overtake events are possible, but the number of events that happen is at most 100.

We need to generalize our solution from test group 1. First, similar to that test group assume that both Anika and the other participants are running very slowly unless they need to overtake. Now each of the other participants can be in one of two states: whether they are ahead or behind Anika within the lap.

Whenever Anika overtakes somebody that was ahead of her within the lap, we just need to change the state of that participant so that they are now behind. We can always make this happen since the other participants can overtake each other in such a way that the participant Anika is going to overtake is the next one on the track.

Whenever Anika overtakes somebody that was already behind of her within the lap, then she must cross the finish line one more time. When this happens, we also need to reset the state of all other participants to being ahead of Anika on the track.

Similar to test group 1, this approach computes a lower bound on the answer, and having the participants only move around as much as is needed for the overtakes we can achieve that bound.

This solution runs in quadratic time since it potentially needs to reset the state of all participants every time.

Test group 4: general case, many events

In this test group all types of overtake events are possible, and the number of events that happen is at most $2 \cdot 10^5$.

We just need to speed up the solution from test group 3 slightly so that resetting the state of all participants is faster. There are several typical approaches to this:

- In addition to storing a boolean to represent the state of each other participant, we also store a vector of indices of participants that are behind Anika. This way we can reset the state of all participants to being ahead of Anika in time proportional to the number of participants that were behind, instead of time proportional to the total number of participants. Since at most one participant switches to be behind Anika at every event, the amortized complexity of this approach is $O(1)$ per event, so the running time is $O(N+Q)$ where Q is the number of events.

- Another option is to store the participants that are behind Anika in a set, and simply clear this set when Anika starts a new lap. The running time is $O(Q)$ if set operations are $O(1)$, or $O(Q \log N)$ if the set operations are $O(\log N)$.

- Yet another option is to store an integer instead of a boolean for each of the other participants: on which lap was the participant behind Anika. This way we do not need to reset anything when Anika starts a new lap, just increment the current lap counter (the answer), so the running time is $O(N+Q)$.