EGOI 2024 Editorial - Circle Passing

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The problem

There are $2N$ people numbered from 0 to $2N-1$ arranged in a circle in such a way that i and $i+1$ are adjacent for all $0 \leq i \leq 2N-2$, and 0 and $2N-1$ are adjacent as well. There are also M friendship relations. A friendship is described by two people k_i ($0 \leq k_i \leq n-1$) and $k_i + n$. In a move, a person can pass the ball either to an adjacent person or to a friend (in case that friendship exists). You will be asked Q queries, and for query j you have to answer the minimum number of passes should be made to send the ball from person x_j to person y_j.

Note that the solutions for test groups build up from one another, so, please read above if you don't understand some details.

General Observations

Simple Paths First, we note that if the ball is sent from person to another, no person should get the ball twice. Otherwise if some person A gets the ball twice, we could skip the passes between the first time A gets the ball and the second time. This would give us a fewer number of passes.

Simple Distance Definition We define the simple distance between person A and person B. The simple distance from person A to person B is the minimum number of passes required to get the ball from person A to person B by only passing the ball to people standing *directly* next to them. In other words, the ball is never passed between two best friends.

Since they stand in a circle, there are only two possible direct paths from person A to person B. We can either go through the left side of person A or through the right side of person A. For one of these paths, $|A-B|$ passes are needed. For the other path, $2N - |A-B|$ are needed since we use all passes that are not in the first path. As we only care about the shortest paths, the simple distance is the minimum of these two values, $\min\{|A-B|, 2N-|A-B|\}$. We refer to this value as the simple distance $sd(A,B)$ between person A and B.

Test group 1: Student starting with the ball has the only friendship

In this test group, there is a single pair of best friends. Further, in every query where the ball needs to get from A to B, person A (who starts with the ball) has a best friend. So, first of all, the possible case is that we just pass the ball to person B by using only direct passes to an adjacent person which takes $sd(A,B)$ as defined above. But now, we also have a single friendship relation we can use. If we use this relationship, we use one move to pass the ball from A to $A+n$, and then use adjacent moves to pass the ball from $A+N$ to B. Note that since the shortest path is simple, we only use this relationship once. In this case, we perform a total of 1 (the first pass from A to $A+N$) + $sd(A+N,B)$ passes. Thus, the answer for a pair is the minimum of these two distances: $min(1+sd(A+N,B), sd(A,B))$.

Test group 2: $N, M, Q \leq 1000$

In this test group, $N, M, Q \leq 1000$. Since N, M and Q are small, we can construct a graph explicitly where each vertex corresponds to a student. Two vertices are connected if the corresponding students can pass the ball to each other, i.e. adjacent students are connected and best friends are connected. For each query, we can perform a breadth-first-search to obtain the distance. This takes $O((N+M) \cdot Q)$ time.

Test group 3: $N \leq 10^7$, $M, Q \leq 1000$

The useful observation we can make here is that we always pass the ball **at most once** using the friendship relations.

Why is that? Because we only care about positions modulo $2N$ (as in positions $mod\ 2N$) (since these are the only positions we have). So, after an even amount of friendship passes (let's say its $2l$), and p adjacent passes, we will have gotten to the position $(A+2l \cdot N+p)$ which is equivalent to $A+p$ modulo $2N$, and in case we do an odd number of friendship passes (let's say its $2l+1$), and p adjacent passes, we will have passed it to the position $A+2l \cdot N+1+p$ which is equivalent to $A+1+p$ modulo $2N$. Note that in the proof we only considered the case where we would increase the position to get to the desired node, but the case where we subtract p is identical so it isn't elaborated to keep the tutorial shorter.

Now, we can iterate over every person with a best friend and see what the shortest path is if we use this friendship relation. For each of the possible paths, we calculate the total number of passes similarly as for the first group. If we use the friendship between some person C and $(C+N) \bmod 2N$, the ball is sent from A to C using adjacent passes, then from C to $(C+N) \bmod 2N$, and from $(C+N) \bmod 2N$ to B which needs $sd(A,C)+1+sd((C+N) \bmod 2N, B)$ passes. Now, we pick the minimum over all possible intermediary Cs. Recall to also consider not using any friendship relations at all which takes $sd(A,B)$ passes.

This takes $O(M \cdot Q)$ running time.

Test group 4: $x_i = 0$ for all i

In this test group we always have to pass the ball from the person at position 0. To solve this test group, we need to make one of the most important observations for the problem. We already know that we need at most one friendship relation. Further, using a friendship relation sooner is always at least as good as using one later. The same proof as in group two applies. As we can see, by performing the moves either sooner or later, we get to the same positions modulo $2N$. Thus, we only need to consider the closest person on the left of A with a best friend and the closest person on the right of A with a best friend. Since A is always 0, these relevant people are the same for every query. We have the same solution as in the previous group but now, instead of checking for m intermediary candidates, we check for only two of them. Again, don't forget about the $sd(A,B)$ case!

Test group 5: General Case

In this test group we need to solve the general case. Here, we just combine all ideas we had in the previous test groups. We know we always need to perform at most one friendship-relation pass, and that this pass has to be via either the closest to the left or via the closest to the right. How do we find the closest to the left and closest to the right people from A? We can use binary search! For this, we keep a sorted array of all people that have a friendship relation. Note that we have to add both k_i and $k_i + N$ to the array. We binary search for the last person with a position at most A, and binary search for the first person with a position of at least A. Additionally, we need to check the first and last positions of people that have a best friend since the positions are cyclic. And then, we can use the same solution as in the fourth test group. This yields a running time of $O(M + Q \log M)$.